

THE KÄHLER METRICS OF CONSTANT SCALAR CURVATURE ON THE COMPLEX TORUS

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Abstract. We study the Dirichlet problem of the Abreu equation. The solutions provide the Kähler metrics of constant scalar curvature on the complex torus.

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One of the central problem in complex geometry is to find certain canonical metrics within a given Kähler class. As examples, the extremal metrics, introduced by E. Calabi, has been studied intensively in the past 20 years. Most extremal metrics are Kähler metrics of constant scalar curvature. There are three aspects of the problem: sufficient conditions of existence, necessary condition of existence and uniqueness. The necessary conditions for the existence are conjectured to be related to certain stabilities. For example, it was first by Tian that gave an analytic "stability" condition which is equivalent to the existence of a Kähler-Einstein metric ([T]). There are many works on this aspects ([T], [D-1],[D-2],[C-T-1]). The uniqueness aspect is completed by Mabuchi ([M]) in the algebraic case and Chen-Tian ([C-T-1]) in general, in the sense that the extremal metric is unique up to the action of holomorphic automorphisms.

On the other hand, there has been not much progress on the existence of extremal metrics or Kähler metrics of constant scalar curvature. One reason is that the equation is highly nonlinear and of 4th order. Our project is to understand this problem on toric varieties following the works of Abreu and Donaldson ([A],[D-3],[D-4]). When studying the equation for Kähler metrics with prescribed scalar curvature on *toric varieties*, one can reduce the equation of complex variables to a real equation on a polytope in \mathbb{R}^n . In [A], using Guillemin's method([G]), Abreu formulates this equation (1.1) which is called the Abreu equation now. Since this is a 4th order equation, the progress on this equation is slow. One of the main result is given by Donaldson. He gives the interior estimates of the equation when $n = 2$ ([D-4]).

In this paper, we study a PDE problem: the Dirichlet problem of the Abreu equation on strictly convex domain with degenerated boundary conditions. We show the interior regularity of the equation. As a corollary, we have constructed abundant Kähler metrics of constant scalar curvature on complex torus $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$. (cf. Remark 1.2 and Corollary 1.3.) The graphs of solution we construct are Euclidean complete, however presumably the associated metrics are not complete on the complex torus.

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The paper is organized as following: in the introduction section we review the equations, formulate our problem and state the main theorem in this paper; the estimates of determinant are given in 2nd section, in particular, Lemma 2.4 is the core lemma of the paper; the rest of paper is devoted to the proof of the main theorem.

1. INTRODUCTION

Given a bounded convex domain $\Omega \subset \mathbb{R}^n$, we study the Abreu equation in this paper

$$\mathcal{S}(u) = K$$

where $K(\xi)$, $\xi = (\xi_1, \dots, \xi_n)$ is some given smooth function defined on an open subset of \mathbb{R}^n containing $\overline{\Omega}$ and $\mathcal{S}(u)$ denotes the expression

$$\mathcal{S}(u) = - \sum \frac{\partial u^{ij}}{\partial \xi_i \partial \xi_j}.$$

The Abreu equation appears in the study of the differential geometry of toric varieties (see [A], [D-1],[D-2]), where K is the scalar curvature of the Kahler metric. The Kahler metric is extremal in the Calabi sense if and only if K is an affine function in ξ . The Abreu equation can be written as (see [D-4] section 2.1)

$$(1.1) \quad \sum_{i,j=1}^n U^{ij} w_{ij} = -K \quad \text{in } \Omega$$

where (U^{ij}) is the cofactor matrix of the Hessian matrix D^2u of the convex function u and $w = \det(u_{ij})^{-1}$.

For any smooth and strictly convex function u on Ω we consider the normal map

$$L_u : \Omega \rightarrow L_u(\Omega) =: \Omega^* \subset \mathbb{R}^n$$

defined by

$$\begin{aligned} (\xi_1, \dots, \xi_n) &\mapsto (x_1, \dots, x_n) \\ x_i &= \frac{\partial u}{\partial \xi_i}. \end{aligned}$$

Then L_u is a diffeomorphism. Define a function $f(x)$ on Ω^* by

$$(1.2) \quad f(x) = \sum_{k=1}^n \xi_k \frac{\partial u}{\partial \xi_k} - u(\xi).$$

f is called the *Legendre transformation* of u . In terms of x_i and $f(x)$ the Abreu equation can be written as

$$(1.3) \quad K = - \sum_{i,j=1}^n f^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \log(\det(f_{kl})),$$

where (f^{ij}) is the inverse of the Hessian (f_{ij}) .

Denote

$$M^* = \{(x, f(x)) | x \in \Omega^*\}.$$

to be the graph of f over Ω^* . If $|\nabla u|_{\partial\Omega} = \infty$, then $f(x)$ is defined on whole \mathbb{R}^n , i.e., M^* is Euclidean complete.

The main result of this paper is the following. The proof is given in §3.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth and strictly convex boundary, K be a smooth function defined on an open subset of \mathbb{R}^n containing $\overline{\Omega}$ such that $K \geq k_o > 0$ for some constant $k_o > 0$. Given a smooth and strictly convex function φ defined on an open subset of \mathbb{R}^n containing $\overline{\Omega}$, then there is a function u such that*

- u is smooth and strictly convex in Ω ;
- on $\partial\Omega$

$$u = \varphi, \quad |\nabla u| = \infty, \quad w = 0;$$

- u solves the Abreu equation

$$\mathcal{S}(u) = K \quad \text{in } \Omega.$$

Remark 1.2. *Let Ω be a convex domain and u be a smooth and strictly convex function that solves the Abreu equation in Ω . Suppose that f is the Legendre transformation of u and $L_u(\Omega) = \mathbb{R}^n$. Then f , as a potential function of the Kähler metric on the complex torus $\mathbb{C}^n/2\pi\sqrt{-1}\mathbb{Z}^n$, gives a T^n -invariant Kähler metric of scalar curvature K .*

Hence we have

Corollary 1.3. *Let u be a solution of the Abreu equation given in Theorem 1.1. Then the Legendre transform function f of u yields a Kähler metric of scalar curvature K . In particular, if K is constant, we construct a T^n -invariant Kähler metric of constant curvature K on the complex torus $\mathbb{C}^n/2\pi\sqrt{-1}\mathbb{Z}^n$.*

Proof. Since $|\nabla u| = \infty$ at boundary, f defines on the whole \mathbb{R}^n . Therefore the claim follows from Remark 1.2. \square

2. ESTIMATES FOR DETERMINANT

In this section we derive some estimates of determinant $\det(D^2u)$. The following two lemmas can be found in [D-4], [T-W-1].

Lemma 2.1. *Suppose that u is a smooth and strictly convex function in Ω satisfying the Abreu equation $\mathcal{S}(u) = K$. If $\det(u_{ij}) > d_1$ near $\partial\Omega$, then*

$$\det(u_{ij}) \geq d_1$$

everywhere in Ω , where

$$d_1 = \left(\frac{4 \max_{\Omega} \{K\} \text{diam}(\Omega)^2}{n} \right)^{-n}.$$

For any $p \in \Omega$ we say that u is normalized at p if

$$u \geq 0, \quad u(p) = 0, \quad \frac{\partial u}{\partial \xi_k}(p) = 0 \quad \forall k = 1, \dots, n.$$

On the other hand, suppose that u is not normalized at p . Let

$$\xi_{n+1} = a \cdot (\xi - p) + b$$

be the support hyperplane of u at p . Set

$$\tilde{u} = u - (a \cdot (\xi - p) + b).$$

Then \tilde{u} is normalized at p . We call \tilde{u} is a *normalization* of u with respect to p .

Suppose that u is normalized at p . For any positive number b we denote

$$\begin{aligned} S_u(p, b) &= \{\xi \in \Omega | u(\xi) < b\}, \\ \bar{S}_u(p, b) &= \{\xi \in \Omega | u(\xi) \leq b\}. \end{aligned}$$

Lemma 2.2. *Suppose that u is a smooth and strictly convex function defined in Ω with $\mathcal{S}(u) = K$. Suppose that u is normalized at p in Ω . If the section*

$$\bar{S}_u(p, C) = \{\xi \in \Omega | u(\xi) \leq C\}$$

is compact and if there is a constant $b > 0$ such that

$$\sum_{k=1}^n x_k^2 \leq b$$

on $\bar{S}_u(p, C)$, then there is a constant $d_2 > 0$ depending on n, C and b such that the estimate

$$\det(u_{ij}) \leq d_2$$

holds in $S_u(p, C/2)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded, normalized convex domain. Thus

$$(2.1) \quad n^{-\frac{3}{2}} D_1(0) \subset \Omega \subset D_1(0).$$

Denote by $\mathcal{F}(\Omega, C)$ the class of convex functions defined on Ω such that

$$\inf_{\Omega} u = 0, \quad u = C \text{ on } \partial\Omega.$$

Lemma 2.3. *Let Ω_k be a sequence of smooth and normalized convex domains, $u_k \in \mathcal{F}(\Omega_k, C)$. Then there are constants $d > 1, b > 0$ independent of k such that*

$$\frac{\sum_i (\frac{\partial u_k}{\partial \xi_i})^2}{(d + f_k)^2} \leq b, \quad k = 1, 2, \dots \quad \text{on } \bar{\Omega}_k.$$

Proof. We may suppose by taking subsequence that Ω_k converges to a convex domain Ω and u_k converges to a convex function u_∞ , locally uniformly in Ω . Obviously, we have the uniform estimate

$$(2.2) \quad \sum \left(\frac{\partial u_k}{\partial \xi_i} \right)^2 (0) \leq 4n^3.$$

For any k , let

$$(2.3) \quad \tilde{u}_k = u_k - \sum \frac{\partial u_k}{\partial \xi_i}(0) \xi_i - u_k(0).$$

Then

$$\tilde{u}_k(0) = 0, \quad \tilde{u}_k(\xi) \geq 0, \quad \tilde{u}_k|_{\partial\Omega_k} \leq C_0,$$

where C_0 is a constant depending only on n . As $B(0, n^{-\frac{3}{2}}) \subset \Omega_k$, we have

$$(2.4) \quad \frac{|\nabla \tilde{u}_k|^2}{(1 + \tilde{f}_k)^2} \leq |\nabla \tilde{u}_k|^2 \leq \frac{C_0^2}{\text{dist}(B(0, 2^{-1}n^{-\frac{3}{2}}), \partial\Omega_k)^2} \leq 4n^3 C_0^2$$

on $B(0, 2^{-1}n^{-\frac{3}{2}})$, where \tilde{f}_k is the Legendre transformation of \tilde{u}_k relative to 0. We discuss three cases.

Case 1. $p \in B(0, 2^{-1}n^{-\frac{3}{2}})$. Following from (2.2), (2.3) and (2.4) we have

$$\frac{|\nabla u_k|^2}{(1 + f_k)^2} \leq 4n^3 (C_0^2 + 1).$$

Case 2. $p \in \partial\Omega_k$, we may suppose that $p = (\xi_1, 0, \dots, 0)$ with $\xi_1 > 0$ by an orthonormal transformation. Then, at p ,

$$C_0 + \tilde{f}_k \geq \tilde{u}_k + \tilde{f}_k = \frac{\partial \tilde{u}_k}{\partial \xi_1} \xi_1.$$

It follows that

$$\frac{\left(\frac{\partial \tilde{u}_k}{\partial \xi_1}\right)^2}{(C_0 + \tilde{f}_k)^2} < \frac{1}{\xi_1^2} < 4n^3.$$

Therefore there exist constants $\tilde{d} > 1$, $\tilde{b} > 0$ depending only on n such that

$$(2.5) \quad \frac{\left(\frac{\partial \tilde{u}_k}{\partial r}\right)^2}{(\tilde{d} + \tilde{f}_k)^2} < \tilde{b},$$

where $\frac{\partial}{\partial r}$ denotes the radial derivative. Note that

$$(2.6) \quad \frac{\partial \tilde{u}_k}{\partial \xi_i} = \frac{\partial u_k}{\partial \xi_i} - \frac{\partial u_k}{\partial \xi_i}(0), \quad \tilde{f}_k = f_k + u_k(0).$$

It follows from (2.3) and (2.4) that

$$\left(\frac{\partial u_k}{\partial r}\right)^2 \leq 2 \left(\frac{\partial \tilde{u}_k}{\partial r}\right)^2 + 8n^3.$$

Then

$$(5.8) \quad \frac{\left(\frac{\partial u_k}{\partial r}\right)^2}{(d' + f_k)^2} < b',$$

for some constants $d' > 1$, $b' > 0$ independent of k . Note that

$$(5.9) \quad |\nabla u_k(p)| = \frac{1}{\cos \alpha_k} \left| \frac{\partial u_k}{\partial r}(p) \right|,$$

where α_k is the angle between vectors $\nabla u_k(p)$ and $\frac{\partial u_k}{\partial r}(p)$. Since $u_k = 1$ on $\partial\Omega_k$, $\nabla u_k(p)$ is perpendicular to the boundary of Ω_k at any $p \in \partial\Omega_k$. As Ω is convex and $0 \in \Omega$, it follows that $\frac{1}{\cos \alpha_k}$ have a uniform upper bound.

Case 3. $p \in \Omega_k^o \setminus B(0, 2^{-1}n^{-\frac{3}{2}})$. Let $F_k = \frac{\sum x_k^2}{(d+f_k)^2}$. We can assume that

$$(2.7) \quad \max_{\partial\Omega_k \cup \bar{B}(0, 2^{-1}n^{-\frac{3}{2}})} F_k < \max_{\Omega_k} F_k.$$

In fact, if (2.7) is not true, then the lemma follows from a direct calculation. Let $p_k^* \in \Omega_k^o \setminus B(0, 2^{-1}n^{-\frac{3}{2}})$ be the point such that

$$F_k(p_k^*) = \max_{\Omega_k} F_k.$$

Then, at p_k^* , $\frac{\partial}{\partial x_i} F_k = 0$. Thus,

$$(2.8) \quad \frac{x_i}{\sum x_j^2} = \frac{\xi_i}{d + f_k}.$$

where $d = d' + C$. Obviously, $d + f_k \geq 1$. Then

$$\frac{\max_{i=1}^n |x_i|}{\sum x_j^2} = \frac{\max |\xi_i|}{d + f_k} \geq \frac{2^{-n}n^{-2}}{d + f_k}.$$

On the other hand,

$$\frac{\max_{i=1}^n |x_i|}{\sum x_j^2} \leq \frac{1}{(\sum x_j^2)^{\frac{1}{2}}}.$$

Noting that $F_k(p) \leq F_k(p_k^*)$, we have

$$F_k(p) \leq 4^n n^4. \quad q.e.d.$$

In the following we prove a stronger estimate than that in Lemma 2.2 which plays an important role in this paper.

Lemma 2.4. *Let u be a smooth and strictly convex function defined in Ω which satisfies the Abreu equation $\mathcal{S}(u) = K$. Suppose that u is normalized at p and the section $\bar{S}_u(p, C)$ is compact. And suppose that there is a constant $b > 0$ such that*

$$(2.9) \quad \frac{\sum x_k^2}{(1+f)^2} \leq b$$

on $\bar{S}_u(p, C)$. Then there is a constant $d_3 > 0$ depending only on n, b and $\frac{1}{C}$, such that

$$(2.10) \quad \exp \left\{ -\frac{4C}{C-u} \right\} \frac{\det(u_{ij})}{(1+f)^{2n}} \leq d_3$$

on $S_u(p, C)$.

Proof. Consider the following function

$$F = \exp \left\{ -\frac{m}{C-u} + L \right\} \frac{1}{w(d+f)^{2n}},$$

where

$$L = \epsilon \frac{\sum x_k^2}{(1+f)^2}.$$

m and ϵ are positive constants to be determined later. Clearly, F attains its supremum at some interior point p^* of $S_u(p, C)$. We have, at p^* ,

$$(2.11) \quad F_i = 0,$$

$$(2.12) \quad \sum u^{ij} F_{ij} \leq 0,$$

where we denote $F_i = \frac{\partial F}{\partial \xi_i}$, $F_{ij} = \frac{\partial^2 F}{\partial \xi_i \partial \xi_j}$, $f_i = \frac{\partial f}{\partial \xi_i}$ and so on. We calculate both expressions (2.11) and (2.12) explicitly:

$$(2.13) \quad -\frac{m}{(C-u)^2} u_i + L_i - 2n \frac{f_i}{1+f} - \frac{w_i}{w} = 0,$$

and

$$(2.14) \quad -\frac{2m}{(C-u)^3} \sum u^{ij} u_i u_j - \frac{mn}{(C-u)^2} + \sum u^{ij} L_{ij} \\ - 2n \frac{\sum u^{ij} f_{ij}}{1+f} + 2n \frac{\sum u^{ij} f_i f_j}{(1+f)^2} + \frac{\sum u^{ij} w_i w_j}{w^2} + K \leq 0.$$

Since

$$f_i = \sum \xi_k u_{ki}, \quad f_{ij} = u_{ij} + \sum \xi_k u_{kij}.$$

Then

$$-2n \frac{\sum u^{ij} f_{ij}}{1+f} = -\frac{2n^2}{1+f} + \frac{2n}{1+f} \sum \xi_k \frac{w_k}{w}.$$

By (2.13)

$$(2.15) \quad -2n \frac{\sum u^{ij} f_{ij}}{1+f} = -\frac{2n^2}{1+f} - \frac{2mn \sum u_i \xi_i}{(C-u)^2(1+f)} + \frac{2n}{1+f} \sum \xi_i L_i - 4n^2 \frac{\sum \xi_i f_i}{(1+f)^2}.$$

Note that

$$\sum \xi_k f_k = \sum u_{kl} \xi_k \xi_l = \sum u^{ij} f_i f_j,$$

Hence

$$\begin{aligned} \frac{2mn}{(C-u)^2(1+f)} \sum u_i \xi_i &= \frac{2mn(u+f)}{(C-u)^2(1+f)} \leq \frac{2mn}{(C-u)^2} \text{Max}\{1, C\}, \\ \frac{2n}{1+f} \sum \xi_i L_i &= \frac{2n}{1+f} \sum \xi_k u_{kj} u^{ji} L_i \leq \frac{1}{2} \frac{\sum u_{kl} \xi_k \xi_l}{(1+f)^2} + 2n^2 \sum u^{ij} L_i L_j \\ &= \frac{1}{2} \frac{\sum u^{ij} f_i f_j}{(1+f)^2} + 2n^2 \sum u^{ij} L_i L_j. \end{aligned}$$

We have

$$(2.16) \quad -2n \frac{\sum u^{ij} f_{ij}}{1+f} \geq -2n^2 - \frac{2mn}{(C-u)^2} \text{Max}\{1, C\} - 2n^2 \sum u^{ij} L_i L_j - \left(4n^2 + \frac{1}{2}\right) \frac{\sum u^{ij} f_i f_j}{(1+f)^2}.$$

Let us calculate the terms $\sum u^{ij} L_i L_j$ and $\sum u^{ij} L_{ij}$. Since

$$L_i = \epsilon \frac{2 \sum x_k u_{ki}}{(1+f)^2} - 2\epsilon \frac{f_i \sum x_k^2}{(1+f)^3},$$

then

$$(2.17) \quad \sum u^{ij} L_i L_j \leq 8\epsilon b \frac{\epsilon \sum u_{kk}}{(1+f)^2} + 8(\epsilon b)^2 \frac{\sum u^{ij} f_i f_j}{(1+f)^2},$$

$$(2.18) \quad \begin{aligned} \sum u^{ij} L_{ij} &= \epsilon \frac{2 \sum u_{kk} + 2 \sum x_k u^{ij} u_{ijk}}{(1+f)^2} - 8\epsilon \frac{\sum x_k f_i u_{kj} u^{ij}}{(1+f)^3} \\ &\quad - 2\epsilon \frac{\sum x_k^2 \sum u^{ij} f_{ij}}{(1+f)^3} + 6\epsilon \frac{\sum x_k^2 \sum u^{ij} f_i f_j}{(1+f)^4} \end{aligned}$$

Applying the Schwarz inequality for each term in (2.18):

$$\begin{aligned} 8\epsilon \frac{\sum x_k f_i u_{kj} u^{ij}}{(1+f)^3} &\leq 16\epsilon^2 \frac{\sum u_{kl} x_k x_l}{(1+f)^4} + \frac{\sum u^{ij} f_i f_j}{(1+f)^2} \leq 16\epsilon b \frac{\epsilon \sum u_{kk}}{(1+f)^2} + \frac{\sum u^{ij} f_i f_j}{(1+f)^2}, \\ 2\epsilon \frac{\sum x_k^2 \sum u^{ij} f_{ij}}{(1+f)^3} &= 2n\epsilon \frac{\sum x_k^2}{(1+f)^3} - 2\epsilon \frac{\sum x_k^2}{(1+f)^3} \sum \xi_k \frac{w_k}{w} \\ &\leq 2n\epsilon \frac{\sum x_k^2}{(1+f)^3} + \frac{1}{8n} \frac{\sum u^{ij} w_i w_j}{w^2} + 8n \left[\epsilon \frac{\sum x_k^2}{(1+f)^2} \right]^2 \frac{\sum u_{ij} \xi_i \xi_j}{(1+f)^2} \\ &\leq 2n\epsilon b + \frac{1}{8n} \frac{\sum u^{ij} w_i w_j}{w^2} + 8n(\epsilon b)^2 \frac{\sum u^{ij} f_i f_j}{(1+f)^2}, \end{aligned}$$

$$\begin{aligned}
2\epsilon \frac{\sum x_k u^{ij} u_{ijk}}{(1+f)^2} &= -2 \frac{\epsilon}{(1+f)^2} \sum x_k \frac{w_k}{w} = -2 \frac{\epsilon}{(1+f)^2} \sum x_l u_{li} u^{ik} \frac{w_k}{w} \\
&\leq \frac{1}{8n} \frac{\sum u^{ij} w_i w_j}{w^2} + 8n \frac{\epsilon^2 \sum u_{kl} x_k x_l}{(1+f)^4} \\
&\leq \frac{1}{8n} \frac{\sum u^{ij} w_i w_j}{w^2} + 8n \epsilon b \frac{\epsilon \sum u_{kk}}{(1+f)^2}.
\end{aligned}$$

Then

$$\begin{aligned}
(2.19) \quad \sum u^{ij} L_{ij} &\geq (2 - 16\epsilon b - 8n\epsilon b) \frac{\epsilon \sum u_{kk}}{(1+f)^2} - \frac{1}{4n} \frac{\sum u^{ij} w_i w_j}{w^2} \\
&\quad - (1 + 8n(\epsilon b)^2) \frac{\sum u^{ij} f_i f_j}{(1+f)^2} - 2nb\epsilon.
\end{aligned}$$

Note that

$$(2.20) \quad \frac{|\sum u^{ij} u_i f_j|}{1+f} = \frac{|\sum x_k \xi_k|}{1+f} = \frac{|u+f|}{1+f} \leq 1.$$

By (2.13), (2.17) and (2.20) we have

$$\begin{aligned}
(2.21) \quad \left(1 - \frac{1}{4n}\right) \frac{\sum u^{ij} w_i w_j}{w^2} &\geq \left(1 - \frac{1}{4n}\right) (1-\delta) \frac{m^2}{(C-u)^4} \sum u^{ij} u_i u_j \\
&\quad + \left(1 - \frac{1}{4n}\right) 4n^2 (1-\delta) \frac{\sum u^{ij} f_i f_j}{(1+f)^2} - \left(\frac{1}{\delta} - 1\right) \sum u^{ij} L_i L_j \\
&\geq \left(1 - \frac{1}{4n}\right) (1-\delta) \frac{m^2}{(C-u)^4} \sum u^{ij} u_i u_j \\
&\quad + \left(1 - \frac{1}{4n}\right) 4n^2 (1-\delta) \frac{\sum u^{ij} f_i f_j}{(1+f)^2} - \frac{8\epsilon b}{\delta} \frac{\epsilon \sum u_{kk}}{(1+f)^2} - \frac{8(\epsilon b)^2}{\delta} \frac{\sum u^{ij} f_i f_j}{(1+f)^2}
\end{aligned}$$

for any small positive number δ . We choose $\epsilon = \frac{1}{8000n^2b}$, $\delta = \frac{1}{200n^2}$, $m = 4C$. Obviously,

$$(2.22) \quad \left(1 - \frac{1}{4n}\right) \left(1 - \frac{1}{24n^2}\right) \frac{m^2}{(C-u)^4} > \frac{2m}{(C-u)^3}.$$

Inserting (2.16), (2.19), (2.19), (2.27) and (2.22) into (2.14), we get

$$\epsilon \frac{\sum u_{kk}}{(1+f)^2} - \frac{3mn}{(C-u)^2} \text{Max}\{1, C\} - 3n^2 + K \leq 0.$$

As

$$\sum u_{ii} \geq n[\det(u_{ij})]^{1/n} = nw^{-1/n}$$

we get

$$\exp \left\{ -\frac{m}{C-u} + \epsilon \frac{\sum x_k^2}{(1+f)^2} \right\} \frac{1}{(1+f)^{2n} w} \leq d$$

for some constant $d > 0$ depending on $n, b, \frac{1}{C}$. Since F attains its maximum at p^* , (2.22) holds everywhere. \square

We remark that $1+f$ in (2.9) can be replaced by any $d+f$ with $d > 0$.

Using the technique of Lemma 2.4 and that of Lemma 4.2 in [L-J], we can prove the following lemma in the case of $n = 2$.

Lemma 2.5. *Let $u \in \mathcal{F}(\Omega, 1)$ with $\mathcal{S}(u) = K$. Suppose that $D_r(0) \subset \Omega$. Then there is a constant $d_3 > 0$ depending only on K_o , b and r , such that*

$$(2.23) \quad (r^2 - \sum \xi_i^2)^2 \frac{\det(u_{ij})}{(d+f)^4} \leq d_3.$$

Proof. Consider the following function

$$F = (r^2 - \theta)^k \exp\{L\} \frac{1}{w(d+f)^4},$$

in $D_r(0)$, where

$$\theta = \sum \xi_i^2, \quad L = \epsilon \frac{\sum x_k^2}{(d+f)^2}.$$

k and ϵ are positive constants to be determined later. Clearly, F attains its supremum at some interior point p^* of $D_r(0)$. At p^* , we have,

$$(2.24) \quad L_i - \frac{4f_i}{d+f} - \frac{w_i}{w} - \frac{k\theta_i}{r^2 - \theta} = 0,$$

and

$$(2.25) \quad \begin{aligned} & \sum u^{ij} L_{ij} - 4 \frac{\sum u^{ij} f_{ij}}{d+f} + 4 \frac{\sum u^{ij} f_i f_j}{(d+f)^2} + K + \frac{\sum u^{ij} w_i w_j}{w^2} \\ & - \frac{k \sum u^{ij} \theta_{ij}}{r^2 - \theta} - \frac{k \sum u^{ij} \theta_i \theta_j}{(r^2 - \theta)^2} \leq 0. \end{aligned}$$

Then as in Lemma ?? we have

$$(2.26) \quad -4 \frac{\sum u^{ij} f_{ij}}{d+f} \geq -8 - 8 \sum u^{ij} L_i L_j - \left(16 + \frac{1}{2}\right) \frac{\sum u^{ij} f_i f_j}{(d+f)^2} - \frac{8k\theta}{r^2 - \theta}.$$

The calculations of the terms $\sum u^{ij} L_i L_j$ and $\sum u^{ij} L_{ij}$ is the same as Lemma 2.4.

By (2.13), (2.17) and (2.20) we have

$$\begin{aligned} \left(1 - \frac{1}{8}\right) \frac{\sum u^{ij} w_i w_j}{w^2} & \geq \left(1 - \frac{1}{8}\right) 16(1 - \delta) \frac{\sum u^{ij} f_i f_j}{(d+f)^2} \\ & - \frac{1}{\delta} \left(\sum u^{ij} L_i L_j + \frac{4k^2 \sum u^{ij} \xi_i \xi_j}{(r^2 - \theta)^2} \right) \\ & \geq -\frac{4k^2 r^2 \sum u^{ii}}{\delta (r^2 - \theta)^2} + \left(1 - \frac{1}{8}\right) 16(1 - \delta) \frac{\sum u^{ij} f_i f_j}{(d+f)^2} \\ & - \frac{8\epsilon b}{\delta} \frac{\epsilon \sum u_{kk}}{(d+f)^2} - \frac{8(\epsilon b)^2}{\delta} \frac{\sum u^{ij} f_i f_j}{(d+f)^2} \end{aligned}$$

for any small positive number δ . We choose $\epsilon = \frac{1}{2000b}$, $\delta = \frac{1}{100}$ and $k = 2$. As (??) we get

$$\epsilon \frac{\sum u_{kk}}{(d+f)^2} - 12 + K - \frac{2 \sum u^{ii} + 8kr^2}{r^2 - \theta} - \frac{4kr^2 \sum u^{ii}}{(r^2 - \theta)^2} - \frac{4k^2 r^2 \sum u^{ii}}{\delta (r^2 - \theta)^2} \leq 0.$$

Denote

$$\Lambda = \frac{2}{r^2 - \theta} + \frac{4kr^2}{(r^2 - \theta)^2} + \frac{4k^2 r^2}{\delta (r^2 - \theta)^2}, \quad \Xi = 12 + |K| + \frac{8kr^2}{r^2 - \theta}.$$

Denote by λ_1, λ_2 the eigenvalues of (u_{ij}) . From (2.27) we have

$$(2.27) \quad \epsilon \frac{\lambda_1 + \lambda_2}{(d+f)^2} - \Xi \leq \Lambda \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right).$$

Then

$$(2.28) \quad \epsilon \frac{\lambda_1 \lambda_2}{(d+f)^2} - \Xi \sqrt{\lambda_1 \lambda_2} \leq \Lambda.$$

where we used the inequality $\lambda_1 + \lambda_2 \geq 2\sqrt{\lambda_1 \lambda_2}$.

Multiplying $e^L(r^2 - \sum \xi_i^2)^2(d+f)^{-2}$ on both sides of (2.28) and applying Schwarz's inequality we get

$$F \leq d$$

for some constant $d > 0$ depending on b, r and K_o . Since F attains its maximum at p^* , (2.23) holds everywhere in $D_r(0)$. \square

By the same calculation of Lemma 2.5 we can obtain

Corollary 2.6. *Let $\Delta \subset \mathbb{R}^2$ be a Delzant polytope. Suppose that $u \in C^\infty(\Delta^\circ)$ satisfies $|S(u)| \leq K_o$. And suppose that there is a constant $b > 0$ such that*

$$(2.29) \quad \frac{\sum x_k^2}{(d+f)^2} \leq b.$$

Then the following estimate holds

$$\frac{\det(u_{ij})}{(d+f)^4} \leq \frac{b_0}{\text{dist}(\xi, \partial\Delta)^4}$$

for some constant $b_0 > 0$ depending only on K_o .

The following lemma is useful for the condition (2.9).

Lemma 2.7. *Let u be a smooth and strictly convex function on Ω and f be its Legendre transform. Suppose that u is normalized at p and satisfies*

$$\frac{\sum x_k^2}{(d+f)^2} \leq b,$$

for some constants $d, b > 0$. For any \tilde{p} , let \tilde{u} be the normalization of u with respect to \tilde{p} . Let \tilde{f} be the Legendre transformation of \tilde{u} . Then there exist constants d' and b' such that

$$\frac{\sum \tilde{x}_k^2}{(d' + \tilde{f})^2} \leq b'.$$

Here $\tilde{x} = \partial\tilde{u}/\partial\xi$.

Proof. Suppose that the support plane at $(\tilde{p}, u(\tilde{p}))$ is

$$\xi_{n+1} = a \cdot (\xi - \tilde{p}) + b.$$

Then $\tilde{u} = u - a \cdot (\xi - \tilde{p}) - b$. By direct computations, we know that

$$\tilde{x}(\xi) = x - a, \quad \tilde{f}(\tilde{x}(\xi)) = f(x(\xi)) - a \cdot \tilde{p} + b.$$

Hence

$$\begin{aligned} \frac{\sum \tilde{x}_k^2}{(d' + \tilde{f})^2} &= \frac{|\tilde{x}|^2}{(d + f)^2} \cdot \frac{(d + f)^2}{(d' + \tilde{f})^2} \\ &\leq \frac{2(|x|^2 + |a|^2)}{(d + f)^2} \cdot \frac{(d + f)^2}{(d' + f - a \cdot \tilde{p} + b)^2} \\ &\leq C(|a|)b = b'. \end{aligned}$$

Here, $C(|a|)$ is a constant depending on $|a|$. d' is chosen so that $d' - a \cdot \tilde{p} + b > d$. \square .

Let $\Delta \in R^n$ be a Delzant polytope (for the definition of Delzant polytope please see [?]). Suppose that Δ has v vertices and d faces of $(n - 1)$ -dimension. Denote by v_A the number of vertices in the $n - 1$ -dimensional face $\ell_A = 0$, and $L = \frac{\min\{v_1, \dots, v_d\}}{v}$. Suppose that

$$\ell_A = \sum a_{Aj} \xi_j - \lambda_A,$$

where $1 \leq A, B, C, \dots \leq d$, $1 \leq i, j, k, \dots \leq n$. Let (y_1, y_2, \dots, y_d) be an affine coordinate system in \mathbb{R}^d .

Lemma 2.8. *Set*

$$P = \{(y_1, \dots, y_d) | y_i > 0 \ \forall i\}.$$

Let $\alpha < \frac{1}{2d}$ be a positive constant, $g(y_1, \dots, y_d)$ be a function defined in P given by

$$g = -(y_1 y_2 \dots y_d)^\alpha.$$

Then g is a smooth and strictly convex function.

Proof. By a direct calculation we have

$$g_{AA} = -\alpha(1 - \alpha) \frac{g}{y_A^2}, \quad A = 1, \dots, d,$$

$$g_{AB} = \alpha^2 \frac{g}{y_A y_B}, \quad \text{for } A \neq B.$$

We claim that the matrix $\left[\left(\frac{1}{2} \delta_{AB} - \alpha \right) \frac{1}{y_A y_B} \right]$ is positive definite. To prove this we should calculate its all principle minors. Let $\{j_1, \dots, j_k\} \subset \{1, \dots, d\}$ such that $j_1 < j_2 < \dots < j_k$. A direct calculation gives us

$$(2.30) \quad \det \left[\left(\frac{1}{2} \delta_{j_k j_l} - \alpha \right) \frac{1}{y_{j_k} y_{j_l}} \right] = \frac{1}{(y_{j_1} y_{j_2} \dots y_{j_k})^2} \frac{1}{2^k} (1 - 2k\alpha) > 0.$$

The claim is proved. It follows that

$$(2.31) \quad \sum_{1 \leq A, B \leq d} g_{AB} h_A h_B \geq \frac{-\alpha g}{2} \sum_{1 \leq A \leq d} \frac{h_A^2}{y_A^2}$$

for any $(h_1, \dots, h_d) \in \mathbb{R}^d$. The lemma is proved. \square .

Let $i : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be given by

$$y_A = \sum_{i=1}^n a_{Ai} \xi_i - \lambda_A.$$

Put $\hat{u}(\xi_1, \dots, \xi_n) = i^* g(y_1, \dots, y_d)$, $\xi \in \Delta$.

Lemma 2.9. *Suppose that $\alpha < \min\{\frac{1}{2d}, \frac{2L}{n}\}$. Then \hat{u} is a smooth and strictly convex function defined on Δ^o , and there is a constant $c > 0$ such that*

$$\det(\hat{u}_{ij}) \geq c \frac{1}{(\ell_1 \dots \ell_d)^{2L-n\alpha}}.$$

Proof. By a linear transformation in $GL(n, \mathbb{Z}^n)$, we may assume that $\ell_i = \xi_i, 1 \leq i \leq n$. Then by (2.31)

$$(2.32) \quad \sum \hat{u}_{ij} b_i b_j \geq \sum \frac{-\alpha g}{2} \left[\sum_{i=1}^n \frac{b_i^2}{\xi_i^2} + \sum_{A=n+1}^d \frac{(\sum a_{Aj} b_j)^2}{\ell_A^2} \right] \geq \sum \frac{-\alpha g}{2} \sum_{i=1}^n \frac{b_i^2}{\xi_i^2}$$

for any $(b_1, \dots, b_n) \in R^n$. It follows that \hat{u} is strictly convex in Δ . By (2.32) we have

$$(2.33) \quad \det(\hat{u}_{ij}) \geq \frac{\alpha^n}{2^n} \frac{|g|^n}{(\xi_1 \dots \xi_n)^2}.$$

For each vertex we have the inequality (2.33). So

$$\det(\hat{u}_{ij})^v \geq \frac{\alpha^{nv}}{2^{nv}} \frac{(\ell_1 \dots \ell_d)^{\alpha nv}}{(\ell_1^{v_1} \dots \ell_d^{v_d})^2}.$$

Therefore there is a constant $c > 0$ such that

$$\det(\hat{u}_{ij}) \geq c \frac{1}{(\ell_1 \dots \ell_d)^{2L-n\alpha}}.$$

The Lemma 2.9 is proved. \square

Lemma 2.10. *Let $\alpha < \min\{\frac{1}{2d}, \frac{2L}{n}\}$. Suppose that $u \in C^\infty(\Delta)$ satisfies the Abreu equation (1.1). Then the following estimate holds*

$$\det(u_{ij}) \geq \frac{1}{b_1 (\ell_1 \dots \ell_d)^\alpha}$$

for some constant $b_1 > 0$.

Proof From the Abreu's equation (1.1) and Lemma 2.9 we can find a constant $b_1 > 0$ such that

$$\sum_{1 \leq i, j \leq n} U^{ij} (w - b_1 (\ell_1 \dots \ell_d)^\alpha)_{ij} > 0.$$

Since $w = 0$ and $\hat{u} = 0$ on $\partial\Delta$, by the maximum principle the lemma follows. \square

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth and strictly convex boundary. Suppose that $u \in C^\infty(\Delta^o)$ satisfies the Abreu equation. Then the following estimate holds*

$$\det(u_{ij}) \geq \frac{1}{b_1 \text{dist}(\bar{\xi}, \partial\Omega)^\alpha}$$

for some constant $b_1 > 0$ and $\alpha < \frac{1}{2(n+1)}$.

Proof. For any point $\bar{\xi} \in \partial\Omega$ we choose coordinates such that $\Omega \subset D$, where

$$D := \{\xi | \xi_1 \geq 0, \dots, \xi_n \geq 0, \xi_1 + \dots + \xi_n \leq a\}$$

and $\bar{\xi} = (0, a_1, \dots, a_n)$. As before we may choose $d > 0$ such that

$$\sum_{1 \leq i, j \leq n} U^{ij} (w - d(\xi_1 \dots \xi_n)^\alpha)_{ij} > 0,$$

where $\ell = a - \sum_{1 \leq k \leq n} \xi_k$, $\alpha = \frac{1}{2(n+1)}$. Since

$$\begin{aligned} w - d(\xi_1 \dots \xi_n \ell)^\alpha &\leq 0 \quad \text{on } \partial\Omega, \\ w - d(\xi_1 \dots \xi_n \ell)^\alpha &= 0 \quad \text{at } \bar{\xi}, \end{aligned}$$

we have

$$w \leq d(\xi_1 \dots \xi_n \ell)^\alpha.$$

It follows that

$$\det(u_{ij}) \geq \frac{1}{d_1 \xi_1^\alpha}$$

for some constant $d_1 > 0$. Since $\bar{\xi}$ is arbitrary, by compactness the lemma is proved. \square

Remark 2.12. *In the proof of Theorem 1.1 we will consider the perturbational Abreu equation*

$$(2.34) \quad \sum U^{ij} w_{ij} = -K \quad \det(u_{kl}) = w^{-1+\theta}$$

where θ is a very small positive number. It is easy to see that the above Lemma 2.1 - Lemma 2.11 remain true for the perturbational Abreu's equation.

3. Proof of Theorem 1.1

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth and strictly convex boundary, φ and ψ be smooth and strictly convex functions defined on an open subset of \mathbb{R}^n containing $\overline{\Omega}$, satisfying

$$C_o^{-1} < \psi < C_o$$

for some constant $C_o > 0$. We first consider the boundary value problem for the perturbational Abreu equation

$$(3.1) \quad \begin{cases} \sum U^{ij} w_{ij} = -K, & \det(u_{kl}) = w^{-1+\theta} & \text{in } \Omega \\ u = \varphi, \quad w = \psi, & & \text{on } \partial\Omega \end{cases}$$

where θ is a very small positive number.

We quote two results from [T-W-2].

Lemma 3.1. *There exists a constant $C_1 > 0$ such that any solution u of (3.1) satisfies*

$$(3.2) \quad C_1^{-1} \leq w \leq C_1$$

$$(3.3) \quad |w(\xi) - w(\xi_o)| \leq C_1 |\xi - \xi_o| \quad \forall \xi \in \Omega, \quad \xi_o \in \partial\Omega,$$

where C_1 depends only on n , $\text{diam}(\Omega)$, $\sup_\Omega K$, $\sup_\Omega |u|$ and C_o .

The proof of this Lemma is the same as Lemma 7.2 in [T-W-2], we only prove (3.2) to indicate that C_1 in (3.2) is independent of θ . Let $F = \log w$. If F attains its minimum at a boundary point, by the boundary condition (3.1) we have $w \geq \inf_{\partial\Omega} \psi$ in Ω . If F attains its minimum at an interior point $\xi \in \Omega$. By a direct calculation, at this point we have

$$0 \leq \sum u^{ij} F_{ij} \leq \frac{-K}{w^\theta} \leq \frac{-k_o}{w^\theta}.$$

where we used $K \geq k_o > 0$. We get a contradiction. Hence $w \geq C_o^{-1}$.

Assume $0 \in \Omega$. Let $F = \log w + \epsilon|\xi|^2$, where ϵ is a constant to be determined. If F attains its maximum at a boundary point, by (3.1) we have $w \leq C_o$. If F attains its maximum at an interior point ξ_0 , Then at this point we have

$$(3.4) \quad 0 = F_i = \frac{w_i}{w} + 2\epsilon\xi_i,$$

$$(3.5) \quad 0 \geq \sum u^{ij} F_{ij} = \frac{-K}{w^\theta} - \frac{\sum u^{ij} w_i w_j}{w^2} + 2\epsilon \sum u^{ii}.$$

Substituting (3.4) into (3.5) and choosing $\epsilon = \frac{1}{5}[\text{diam}(\Omega)]^{-2}$ we have

$$0 \geq \frac{-K}{w^\theta} + \sum (2\epsilon - 4\epsilon^2 \xi_i^2) u^{ii} \geq \frac{-K}{w^\theta} + \epsilon u^{ii} \geq \frac{-K}{w^\theta} + \epsilon u^{ii} \geq \frac{-K}{w^\theta} + \epsilon w^{\frac{1-\theta}{n}},$$

where we used $\sum u^{ii} \geq \det(u_{ij})^{-1/n}$. Noting that $F \leq F(\xi_0)$, then

$$w \leq e^2 [C_o]^{(n-1)\theta} [5 \max |K| \text{diam}(\Omega)^2]^n.$$

Hence as $\theta \leq 1$, C_1 is independent of θ . (3.2) is proved. \square

Proposition 3.2. *The boundary value problem (3.1) admits a solution $u^{(\theta)} \in C^\infty(\bar{\Omega})$.*

Please refer to Theorem 1.2 and the remark in the end of [T-W-2] for the proof.

Letting $\theta \rightarrow 0$, we have

Proposition 3.3. *There is a convex function $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ with $\det(u_{ij}) \in C^0(\bar{\Omega})$ such that*

$$(3.6) \quad \begin{cases} \sum_{1 \leq i, j \leq n} U^{ij} w_{ij} = -K, & \det(u_{kl}) = w^{-1} & \text{in } \Omega \\ u = \varphi, \quad w = \psi, & & \text{on } \partial\Omega \end{cases}$$

Proof. By Lemma 3.1 and the comparison principle of Monge-Ampere equations it follows that $|u^{(\theta)}|$ is uniformly bounded. By (3.2) and a result of Caffarelli([C1]), u is strictly convex in Ω . Applying the Caffarelli-Gutierrez theory for linearized Monge-Ampere equations (cf. [CG]) we obtain $\det(u_{ij}) \in C^\alpha(\Omega)$, for some $\alpha \in (0, 1)$. By Caffarelli's $C^{2,\alpha}$ estimates for Monge-Ampere equation ([C2]) we have

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C_2.$$

Following from the standard elliptic regularity theory we have $\|u\|_{W^{4,p}(\Omega)} \leq C$. By Sobolev embedding theorem we have

$$\|u\|_{C^{3,\alpha}(\Omega)} \leq C_2 \|u\|_{W^{4,p}(\Omega)}.$$

Using standard bootstrap skill we conclude that

$$\|u\|_{C^\infty(\Omega)} \leq C_3.$$

By (3.3) and the bounds of $|u^{(\theta)}|$, we have $u \in C^0(\bar{\Omega})$ and $\det D^2 u \in C^0(\bar{\Omega})$. The Proposition is proved. \square .

Remark 3.4. In [CG] Caffarelli-Gutierrez proved a Hölder estimate of $\det(u_{ij})$ for homogeneous linearized Monge-Ampere equation assuming that the Monge-Ampère measure $\mu[u]$ satisfies some condition, which is guaranteed by (3.2). When $f \in L^\infty$, following their argument one can obtain Hölder continuity of $\det(u_{ij})$.

The rest of the section is to prove Theorem 1.1. We consider the boundary value problem

$$(3.7) \quad \begin{cases} \sum U^{ij} w_{ij} = -K, & \det(u_{kl}) = w^{-1} & \text{in } \Omega \\ u = \varphi, & w = t, & \text{on } \partial\Omega \end{cases}$$

where $t > 0$. We have a family of solutions $u^{(t)}$. Let $t \rightarrow 0$. We are going to prove that:

- for any compact set $D \subset \Omega$ there is a subsequence $u^{(t_i)}$, which uniformly C^∞ -converges on D ; set the limit to be u ;
- u is smooth, strictly convex and satisfies the Abreu equation (1.2);
- on $\partial\Omega$

$$u = \varphi, \quad |\nabla u| = \infty, \quad w = 0.$$

We divide the proof into 4 steps. Each subsection consists of one step.

3.1. Step One. We first give a uniform estimate for $\max_\Omega \{u^{(t)}\} - \min_\Omega \{u^{(t)}\}$. In the following calculation we will omit the index (t) to simplify notations. By adding a constant we may assume that $\max_\Omega \{u\} = 0$. Then $u \leq 0$. We show that

Lemma 3.5. *There is a constant $d_4 > 0$ independent of t such that $|u^{(t)}| \leq d_4$.*

Proof. We have

$$(3.8) \quad \begin{aligned} - \int_\Omega K(u - \varphi) &= \int_\Omega \sum U^{ij} w_{ij} (u - \varphi) = - \int_\Omega \sum U^{ij} (u - \varphi)_i w_j \\ &= - \int_{\partial\Omega} w \sum U^{ij} (u - \varphi)_i \gamma_j + \int_\Omega w \sum U^{ij} (u - \varphi)_i \gamma_j \\ &\leq - \int_{\partial\Omega} w \sum U^{ij} (u - \varphi)_i \gamma_j + n|\Omega|, \end{aligned}$$

where γ denotes the unit outward normal vector on $\partial\Omega$. We now calculate the integral $\int_{\partial\Omega} w \sum U^{ij} (u - \varphi)_i \gamma_j$. For any boundary point $\bar{\xi}$, by choosing coordinates, we may assume that

$$\gamma = (0, \dots, 0, -1).$$

We have

$$\sum U^{ij} (u - \varphi)_i \gamma_j = (u - \varphi)_n U^{nn}.$$

Near $\bar{\xi}$ the boundary $\partial\Omega$ can be given by a smooth and strictly convex function

$$\xi_n = h(\xi_1, \dots, \xi_{n-1})$$

with

$$h(0) = 0, \quad \frac{\partial h}{\partial \xi_k}(0) = 0 \quad \forall k = 1, \dots, n-1.$$

We may write $\varphi = u(\xi_1, \dots, \xi_{n-1}, h(\xi_1, \dots, \xi_{n-1}))$ on the boundary. By a direct calculation we have

$$u_{kl} = \varphi_{kl} + u_n h_{kl}.$$

So

$$(3.9) \quad U^{nn} = \det(u_{kl})|_{k,l=1}^{n-1} \leq a \left(\left| \frac{\partial u}{\partial \xi_n} \right|^{n-1} + 1 \right),$$

where a is a constant depending on φ and h .

We estimate $|\frac{\partial u}{\partial \xi_n}|$. By the same argument of Lemma 3.1, $\det(u_{ij})$ attains its maximum on the boundary. Therefore there is a uniform estimate

$$\det(u_{ij}) \leq \frac{1}{t}.$$

Let $\bar{\xi} \in \partial\Omega$ be an arbitrary point. As $\partial\Omega$ is smooth and strictly convex, by an affine transformation (see [L-S-C]) we may assume that $u(\bar{\xi}) = \min_{\partial\Omega}\{u\}$. Moreover, there is point ξ_o and a constant $R > 0$ independent of $\bar{\xi}$ such that $\Omega \subset B_{\xi_o}(R)$, and Ω tangent to $B_{\xi_o}(R)$ at $\bar{\xi}$. Let \hat{u} be the function defined by

$$\hat{u} = \frac{1}{2} \sqrt[n]{\frac{1}{t}} (|\xi - \xi_o|^2 - R^2) + u(\bar{\xi}).$$

We have

$$\det(u_{ij}) \leq \det(\hat{u}_{ij}), \quad u|_{\partial\Omega} \geq \hat{u}|_{\partial\Omega}.$$

By the maximum principle, we have $\hat{u} \leq u$ in Ω . As $\hat{u}(\bar{\xi}) = u(\bar{\xi})$, we get

$$(3.10) \quad |\nabla u| \leq R \sqrt[n]{\frac{1}{t}}.$$

It follows from (3.8)-(3.10), we have

$$(3.11) \quad \int_{\partial\Omega} w \sum U^{ij} (u - \varphi)_i \gamma_j \leq C_2$$

for some constant $C_2 > 0$. Therefore

$$- \int_{\Omega} K(u - \varphi) \leq C_3$$

for some constant $C_3 > 0$. Now suppose that $|u|$ attains its maximum at $\hat{\xi}_o$. Let $Cone(p)$ be the cone with base $\partial\Omega$ and vertex at $p = (\hat{\xi}_o, u(\hat{\xi}_o))$. As $u - \varphi \leq 0$ and $-K \leq -k_o$, we have

$$C_3 \geq - \int_{\Omega} K(u - \varphi) \geq k_o \int (\varphi - u).$$

Therefore

$$C_4 := \frac{C_3}{k_o} + \int (-\varphi) \geq \int (-u) \geq Vol(Cone(p)) = \frac{1}{n+1} \max_{\Omega} \{|u|\} |\Omega|.$$

Hence there is a constant d_4 independent of t such that

$$|u^{(t)}| \leq d_4.$$

We finish the proof. \square .

For any compact set $D \subset \Omega$,

$$|\nabla u^{(t)}| \leq \frac{d_4}{dist(D, \partial\Omega)}.$$

It follows that there is a convex function u defined in Ω such that for any compact set $D \subset \Omega$ there is a subsequence $u^{(t_i)}$ converging uniformly on D to u . We denote the graph of u by

$$M = \{(\xi, u(\xi)) | \xi \in \Omega\}.$$

3.2. Step Two. In this step, we prove

Lemma 3.6. *For any point $\bar{p} = (\bar{\xi}, \varphi(\bar{\xi}))$ with $\bar{\xi} \in \partial\Omega$,*

$$|\nabla u^{(t)}|(\bar{p}) \rightarrow \infty$$

as $t \rightarrow 0$.

Proof. Suppose that there is a point $\bar{p} = (\bar{\xi}, \varphi(\bar{\xi}))$, where $\bar{\xi} \in \partial\Omega$, and a subsequence t_i such that $\lim_{i \rightarrow \infty} |\nabla u^{(t_i)}|(\bar{p}) = a$ for some constant $a \geq 0$. Denote $u^{(i)} = u^{(t_i)}$. $u^{(i)}$ will locally uniformly converges to a convex function u . Set

$$K_o = \frac{5 \max_{\Omega} \{K\}}{4d_1}.$$

Since $\partial\Omega$ is smooth and strictly convex, by an affine transformation (see [L-S-C]) we may assume that

- $$u^{(i)}(\bar{\xi}) = \max_{\partial\Omega} \{u^{(i)}\} = -\frac{n}{K_o} t_i, \quad \bar{\xi} = (0, \dots, 0),$$
- the equation of the tangent hyperplane of $\partial\Omega$ at $\bar{\xi}$ is $\xi_1 = 0$ and $\Omega \subset \{\xi_1 > 0\}$.

Let $\Delta(c, r)$ be a domain defined by

$$\Delta(c, r) = \{(\xi_1, \dots, \xi_n) | c(\xi_2^2 + \dots + \xi_n^2) < \xi_1 < r.\}$$

Since $\partial\Omega$ is smooth, we may choose some constant c and small r (depending on Ω) so that $\Delta(c, r) \subset \Omega$. Then $u^{(i)} < 0$ and there is a constant $b > 0$ such that for any small $\epsilon > 0$

$$(3.12) \quad |u^{(i)}| \leq b\xi_1$$

on the part $\{\xi_1 \geq \epsilon\} \cap \Delta(c, r)$ for large i . Denote the graph of $u^{(k)}$ by

$$M^{(k)} = \left\{ (\xi, u^{(k)}(\xi)) | \xi \in \Omega \right\},$$

We have

$$\sum U^{(k)ij} \left[w^{(k)} + \frac{K_o}{n} u^{(k)} \right]_{ij} = -K + K_o \det(u_{ij}^{(k)}) \geq 0 \quad \text{in } \Omega,$$

$$w^{(k)} + \frac{K_o}{n} u^{(k)} \leq 0 \quad \text{on } \partial\Omega,$$

for k large enough. By the maximum principle we get

$$-w^{(k)} \geq \frac{K_o}{n} u^{(k)} \quad \text{on } \bar{\Omega},$$

i.e.,

$$w^{(k)} \leq \frac{K_o}{n} |u^{(k)}| \quad \text{in } \Omega.$$

Then

$$(3.13) \quad \det(u_{ij}^{(k)}) \geq \frac{n}{bK_o\xi_1}$$

on $\{\xi_1 \geq \epsilon\} \cap \Delta(c, r)$ for k large enough. We construct a new function \hat{u} as following. We define \hat{u} in $\Delta(c, r)$:

$$\hat{u} = -(\xi_1 - c(\xi_2^2 + \dots + \xi_n^2)) (-\log \xi_1)^\alpha + \xi_1 (-\log r)^\alpha,$$

where $\alpha > 0$ is a small positive number. It is easy to check that

$$\hat{u}(\bar{\xi}) = 0,$$

$$\hat{u} \geq u^{(k)} \quad \text{on } \partial\Delta(c, r) \bigcap \{\xi_1 \geq \epsilon\},$$

$$(3.14) \quad \frac{\partial \hat{u}}{\partial \xi_1}(\bar{\xi}) = -(-\log \xi_1)^\alpha + (-\log r)^\alpha + \frac{\alpha}{(-\log \xi_1)^{1-\alpha}} - \frac{\alpha c(\xi_2^2 + \dots + \xi_n^2)}{(-\log \xi_1)^{1-\alpha} \xi_1}.$$

We now calculate $\det(\hat{u}_{ij})$: Let $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \Delta$. By taking an orthogonal transformation we may assume that $\xi = (\xi_1, \xi_2, 0, \dots, 0)$. By a direct calculation we have, at ξ ,

$$\hat{u}_{11} = \frac{\alpha}{(-\log \xi_1)^{1-\alpha} \xi_1} + \alpha \frac{c\xi_2^2}{(-\log \xi_1)^{1-\alpha} \xi_1^2} + \frac{\alpha(1-\alpha)}{(-\log \xi_1)^{2-\alpha} \xi_1} - \alpha(1-\alpha) \frac{c\xi_2^2}{(-\log \xi_1)^{2-\alpha} \xi_1^2},$$

$$\hat{u}_{12} = \hat{u}_{21} = -\frac{2c\alpha\xi_2}{(-\log \xi_1)^{1-\alpha} \xi_1},$$

$$\hat{u}_{kk} = 2c(-\log \xi_1)^\alpha \quad \forall \quad k \geq 2, \quad \hat{u}_{ij} = 0 \quad \text{for other cases.}$$

We choose $\alpha = \frac{1}{2(n+1)}$. Obviously, (\hat{u}_{ij}) is positive definite for $\xi_1 < e^{-1}$, and

$$(3.15) \quad \det(\hat{u}_{ij}) \leq \frac{\alpha(2c)^{n-1}}{(-\log \xi_1)^{1-n\alpha} \xi_1}.$$

It is easy to see that

$$\det(u_{ij}^{(k)}) \geq \det(\hat{u}_{ij}^{(k)}) \quad \text{on } \Delta(c, r) \bigcap \{\xi_1 \geq \epsilon\}$$

for k large enough. By the maximum principle, we have

$$\hat{u} \geq u^{(k)} \quad \text{on } \Delta(c, r) \bigcap \{\xi_1 \geq \epsilon\}.$$

Let $k \rightarrow \infty$, $\epsilon \rightarrow 0$, we get

$$(3.16) \quad \hat{u} \geq u \quad \text{in } \Delta(c, r).$$

From (3.12), (3.13) and (3.16) we get a contradiction. The lemma is proved. \square

Using the same method we can also prove

Lemma 3.7. *For any point $\xi \in \Omega$ and any support hyperplane H of M at $p = (\xi, u(\xi))$, $\text{dist}(H, \partial M) > 0$.*

Denote by $f^{(t)}(x)$ the Legendre transformation functions of $u^{(t)}$. For any large $R > 0$, by Lemma 3.6, there is a $t_0 > 0$ such that $f^{(t)}$ is defined on the disk $B_0(R)$ for $0 < t < t_0$. Since $|\nabla f^{(t)}|$ are uniformly bounded by $\text{diam}(\Omega)$, there is a subsequence $f^{(t_i)}$ converging to a convex function f defined on $B_0(R)$. Let $R \rightarrow \infty$. By choosing subsequences we conclude that there exists a convex function f defines on whole \mathbb{R}^n such that $f^{(t)}$ locally uniformly converges to f .

3.3. **Step Three.** We first prove

Lemma 3.8. *We assert that*

- *For any point $\xi \in \Omega$ and any support hyperplane H of M at $p = (\xi, u(\xi))$*

$$\dim(M \cap H) \leq n - 1;$$

- *For any point $\xi \in \Omega$ and any ball $B_\xi(\delta') \subset \Omega$ with radius δ' around ξ , there exists a point $\xi_o \in B_\xi(\delta')$ such that u has second derivatives and strictly convex at ξ_o .*

Proof. The first claim follows from the weakly continuous of Monge -Ampere measure. We prove the second one. Since u is convex, it has second order derivatives almost everywhere. Let $G \subset B_\xi(\delta')$ be the set where u has second order derivatives. Then $|B_\xi(\delta') - G| = 0$. Let O be an open subset of $B_\xi(\delta')$ such that $B_\xi(\delta') - G \subset O$ with $|O| \leq \epsilon'$. We choose ϵ' so small that $|B_\xi(\delta') - O| > \frac{1}{2}|B_\xi(\delta')|$. By the weak convergence of the Monge-Ampere measure we have

$$(3.17) \quad \int_{B_\xi(\delta') - O} \det(u_{kl}) > \frac{1}{2}d_1|B_\xi(\delta')|.$$

We claim that there exists a point $\xi^o \in B_\xi(\delta') - O$ such that

$$\det(u_{kl})(\xi^o) \geq \frac{d_1|B_\xi(\delta')|}{2|B_\xi(\delta') - O|}.$$

Otherwise, we would have

$$\int_{B_\xi(\delta') - O} \det(u_{kl}) \leq \frac{1}{2}d_1|B_\xi(\delta')|,$$

which contradicts to (3.17). As u has second order derivatives at ξ_o , there is a constant B such that $\lambda_k \leq B$ for all k , where λ_k denote the eigenvalues of $u_{kl}(\xi_o)$. Denote by λ_1 the least eigenvalue of u_{kl} . Then

$$\frac{d_1}{4} < \frac{d_1|B_\xi(\delta')|}{2|B_\xi(\delta') - O|} < \lambda_1 B^{n-1},$$

which implies that u is strictly convex at p_o . \square .

Without loss of generality we assume that u has second order derivatives and strictly convex at 0, and

$$u \geq 0, \quad u(0) = 0.$$

Then there is a positive number $a > 0$ such that

- $\partial M \cap \{\xi_{n+1} = a\} \neq \emptyset$
- $\partial M \cap \{\xi_{n+1} = a - \epsilon\} = \emptyset$ for any $\epsilon > 0$.

Lemma 3.9. *u is smooth and strictly convex in $S_u(0, a)$.*

Proof. Suppose that $u^{(i)}$ locally uniformly converges to u . Since u is strictly convex at 0, there is a small positive number ϵ'' and $b' > 0$ such that $\bar{S}_{u^{(i)}}(0, \epsilon'')$ is compact and

$$\sum \left(\frac{\partial u^{(i)}}{\partial \xi_k} \right)^2 \leq b' \quad \text{in } S_{u^{(i)}}(0, \epsilon'')$$

for large i . By Lemma 2.1 and Lemma 2.2 we have uniform estimates for $\det(u_{kl}^{(i)})$ on $S_{u^{(i)}}(0, \frac{1}{2}\epsilon'')$ both from above and below. Then we use the Caffarelli-Gutierrez

theory and the Caffarelli-Schauder estimate to conclude that $\{u^{(i)}\}$ smoothly converges u . Therefore, u is a smooth and strictly convex function in $S_u(0, \frac{1}{2}\epsilon'')$ and u satisfies the Abreu equation $\mathcal{S}(u) = K$.

Let $f^{(i)}(x)$ be the legendre transformations of $u^{(i)}$. Then $f^{(i)}$ locally uniformly converge to a convex function $f(x)$ defined on the whole \mathbb{R}^n . Furthermore, in a neighborhood of 0, $f(x)$ is a smooth strictly convex function such that its Legendre transform u satisfies the Abreu equation. Denote

$$\widetilde{M}^{(\infty)} = \{(x_1, \dots, x_n, f(x))\}.$$

By the convexity of $f^{(i)}$ there is a constant $b'' > 0$ such that

$$\frac{\sum x_k^2}{(1 + f^{(i)})^2} \leq b$$

for any i .

Let $\Gamma \subset S_u(0, a)$ be the set of points where u is smooth and strictly convex. By the same argument given above, we know that $u^{(i)}$ smoothly converges to u on Γ and Γ is open. If $\Gamma = S_u(0, a)$, the lemma is proved. Otherwise, there is a constant c with $0 < c < a$ such that $S_u(0, c) \subset \Gamma$ but $\bar{S}_u(0, c)$ is not. When this is the case, we choose a point $\bar{\xi} \in \bar{S}_u(0, c) \setminus \Gamma$. we may choose a point $\xi_o \in S_u(0, c)$ close to $\bar{\xi}$ and a small $\delta > 0$ such that

$$\bar{\xi} \in S_u(\xi_o, \delta) \subset S_u(0, a).$$

It is then again that we repeat the above argument to show that u is smooth and strictly convex in the section $S_u(\xi_o, \delta)$: let $\hat{u}^{(i)}$ be the normalization of $u^{(i)}$ at ξ_o ; let $\hat{x} = \partial \hat{u}^{(i)} / \partial \xi$ and $\hat{f}^{(i)}$ be the Legendre transform of $\hat{u}^{(i)}$; then by Lemma 2.7

$$\frac{|\hat{x}|^2}{(d + \hat{f}^{(i)})^2} \leq b.$$

Then applying Lemma 2.4 we get uniform bounds of $\det(u_{ij}^{(i)})$ at $S_{\hat{u}^{(i)}}(\xi_o, \delta')$, $\delta' < \delta$ for large i ; therefore $\hat{u}^{(i)}$, and so $u^{(i)}$, smoothly converges in this domain. This contradicts to the assumption that $\bar{\xi} \notin \Gamma$. \square

We are going to prove that u is smooth and strict convex in Ω . If not, then there is a open set $U \subset \bar{U} \subset \Omega$ such that

- u is smooth and strictly convex in U ;
- there is a point $\bar{q} = (\bar{\xi}, u(\bar{\xi}))$, $\bar{\xi} \in \bar{U}$, and a line segment L such that $\bar{p} \in L \subset M$.

By Lemma 3.5 for any support hyperplane P of M at \bar{q} we have $\text{dist}(\partial M, H) > 0$. We can choose a point $\xi_o \in U$, very close to $\bar{\xi}$, and a positive number a_1 such that

$$\bar{\xi} \in S_u(\xi_o, P, \frac{1}{2}a_1), \text{ and, } \partial M \cap \{\xi_{n+1} = u(\xi_o) + P(\xi - \xi_o) + b\} = \emptyset$$

for any $b \leq a_1$, where $(P, -1)$ is the normal vector of M at $(\xi_o, u(\xi_o))$ and

$$S_u(\xi_o, P, \frac{1}{2}a_1) := \{\xi \in \Omega | u(\xi) < u(\xi_o) + P(\xi - \xi_o) + \frac{1}{2}a_1\}.$$

By Lemma 3.9, we know that u is smooth and strictly convex in $S_u(\xi_o, a_1)$, we get a contradiction. It follows that u is smooth and strictly convex in Ω .

3.4. Step Four. Since u is smooth and strictly convex in Ω , it can be extended over $\partial\Omega$. Let φ' be the function. Then obviously, on $\partial\Omega$, $\varphi' \leq \varphi$. We prove that

Lemma 3.10. $\varphi' = \varphi$.

Proof. Suppose that there is a point $\bar{\xi} \in \partial\Omega$ such that $\varphi'(\bar{\xi}) < \varphi(\bar{\xi})$. Without loss of generality, we assume that

- $\varphi(\bar{\xi}) = 1$, $\varphi'(\bar{\xi}) = 0$,
- $\bar{\xi} = (0, \dots, 0)$, and the equation of the tangent hyperplane of $\partial\Omega$ at $\bar{\xi}$ is $\xi_1 = 0$ and $\Omega \subset \{\xi_1 > 0\}$,
- restricting to $\{\xi_1 \leq \epsilon\} \cap \partial\Omega$, $\varphi' < \frac{1}{10}$.

We construct a function \tilde{u}

$$\tilde{u} = 2u - b\xi_1 + \frac{1}{2}.$$

By choosing b large, we have $\tilde{u} + \frac{1}{3} \leq u$ on $\partial\Delta'$, where

$$\Omega' := \{\xi \in \Omega \mid \xi_1 \leq \epsilon\}.$$

For any positive number $\delta > 0$, let $D_\delta = \{\xi \in \Omega' \mid \text{dist}(\xi, \partial\Omega') \geq \delta\}$. Then for δ small enough and i large enough we have

$$\tilde{u} < u \quad \text{on } \partial D_\delta,$$

$$\det(\tilde{u}_{kl}) > \det(u_{kl}^{(i)}).$$

It follows that $u^{(i)} \geq \tilde{u}$ on \overline{D}_δ . Letting $\delta \rightarrow 0$, $i \rightarrow \infty$ we get $\tilde{u} \leq u$ in Δ' . But $\tilde{u}(\bar{\xi}) = \frac{1}{2} > \varphi'(\bar{\xi})$, and both \tilde{u} and u is smooth in the interior of Δ' , we get a contradiction. So $\varphi' = \varphi$ on $\partial\Omega$. The claim $w = 0$ on $\partial\Omega$ follows from Lemma 2.11. The claim $|\nabla u| = \infty$ follows from the fact that f is defined on the whole \mathbb{R}^n .

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